Geometry 2023
Exam, Tuesday 11 April, 08:30-10:30

- Below you can find the exam questions. There are 4 questions summing up to 85 points, you get extra 15 points for a clear writing of solutions.
- You may consult two A4 pages handwritten or typeset by you for formulas, results, etc treated in this course. Other materials are not allowed.
- When handing in your solutions, please do not forget to write your name and student number on the envelope. Good luck!

QUESTIONS

1. **[5+15 = 20 pts]** Consider the following parametric equation of a tractrix

$$\gamma = \left(t - \tanh t, \frac{1}{\cosh t}\right) : \mathbb{R} \to \mathbb{R}^2$$

   i) Determine whether $\gamma = \gamma(t)$ is an everywhere regular curve,

   ii) Compute the evolute of γ and state its relation to Huygens's principle.

2. **[10+10+15 = 35 pts]** Consider the torus of revolution T^2 in $\mathbb{R}^3$ given by parametric equations

$$\begin{aligned} x &= (2 + \cos\psi)\cos\varphi, \\ y &= (2 + \cos\psi)\sin\varphi, \\ z &= \sin\psi, \quad (\varphi, \psi) \in [0, 2\pi]^2. \end{aligned}$$

   i) Prove that the meridians $\varphi = \text{const}$ and the two parallels $\psi = 0, \pi$ are geodesics of T^2.

   ii) Prove that there exists a geodesic triangle on the torus T^2 for which the angle sum is a) strictly greater than π and b) strictly less than π.

   iii) Using part i), subdivide T^2 into 4 geodesic triangles. Deduce that $\frac{1}{2\pi}\int_{T^2} K \mathrm{d}S = 0$, where $K : T^2 \to \mathbb{R}$ is the Gaussian curvature of T^2 and $\mathrm{d}S$ is the area element.

Hint: local Gauss-Bonnet theorem. You may use without proof that every pair of points on T^2 is connected by a geodesic.

see next page

3 [8+7 = 15pts] Consider the affine 3-space $\mathbb{Q}^3$, where $\mathbb{Q}$ is the field of rational numbers. Let A, B, C, and A', B', C' be 6 distinct points in $\mathbb{Q}^3$. Assume that the line AB is parallel to $A'B'$, BC is parallel to $B'C'$, and AC is parallel to $A'C'$. Prove that

i) If two of the lines AA', BB', and CC' intersect, then these three lines are concurrent (i.e., intersect at a single point),

ii) If two of the lines AA', BB', and CC' are parallel, then all three lines are parallel.

4 [15pts] Let (e_1, e_2) be a basis of $\mathbb{C}^2$ and $P(\mathbb{C}^2)$ be realised as the projective completion of the affine line $\{ze_1 + e_2 \mid z \in \mathbb{C}\}$ in $\mathbb{C}^2$. Similarly, let (e^1, e^2) be the dual basis of $(\mathbb{C}^2)^*$ and $P((\mathbb{C}^2)^*)$ be the projective completion of the affine line $\{ve^1 + e^2 \mid v \in \mathbb{C}\}$. Let $A: \mathbb{C}^2 \to \mathbb{C}^2$ be a linear isomorphism and $A^*: (\mathbb{C}^2)^* \to (\mathbb{C}^2)^*$ be its dual. Let f and f^* be the corresponding projective transformations of $P(\mathbb{C}^2)$ and $P((\mathbb{C}^2)^*)$. Compute f^* in the coordinate v when $f = z + 1$ is a translation.

Hint. Recall that in matrix form, A^* is given by the transpose of A.

End of exam